Unsteady Boundary Layers with Flow Reversal and the Associated Heat-Transfer Problem

J. C. T. Wang* and S. F. Shen†
Cornell University, Ithaca, N. Y.

Two specific examples of nonseparating unsteady boundary layers with local flow reversal, and their associated thermal boundary layers, are constructed from the class of semisimilar solutions. The governing equations are so written that proven numerical schemes for steady boundary layers can be applied without modification. The results show no drastic change of heat transfer at flow reversal, invalidating the usual concept of Reynolds analogy. The structure of the equations meanwhile suggests a separation criterion which clearly accounts for the unsteady effects.

I. Introduction

FOR the two-dimensional steady boundary layer over a fixed wall and with a prescribed adverse pressure gradient, numerical solutions indicate the presence of a singularity at the streamwise location where the wall shear vanishes. The nature of the singularity was identified by Goldstein[1] by an asymptotic expansion in the neighborhood of the point of zero shear along the wall. The condition of vanishing wall shear has also traditionally been used to identify the separation point. Vanishing of the wall shear indicates incipient flow reversal which, however, is not synonymous with separation. Rather, the key feature is the breakdown of the boundary-layer equation, implying that the exterior inviscid flow downstream could no longer adhere to the wall in the limit of infinite Reynolds number. It may be noted that the steady boundary-layer equation can be integrated beyond the point of zero wall shear by prescribing the displacement thickness[2] or the wall-shear variation,[3] instead of the pressure gradient. The boundary-layer equation is then capable of describing unseparated steady flow that contains a recirculating bubble.

In an unsteady boundary layer, the vanishing of the wall shear does not signify a singular behavior of the boundary-layer solution.[4-6] Consequently, it no longer serves as a criterion for separation. The question of why the steady boundary-layer equation is singular at the point of zero wall shear while the unsteady boundary-layer equation is not has been given some insight by Shen and Nenni.[7] From a different viewpoint, the unsteady boundary-layer equation as is, i.e., *with prescribed pressure distribution*, must be able to describe some cases of unseparated flows with recirculating bubble. Indeed, Telionis et al.[8] and Phillips and Ackerberg[9] have obtained such numerical solutions and found no difficulty in passing through the point of zero wall shear. However, they need special differencing schemes inside the reversed flow region, and these schemes are constructed on physical arguments. In the more restricted class of semisimilar unsteady boundary layers, the equations can be reduced to resemble formally those for a steady boundary layer. Some interesting examples of solution are given by Tani[10] and Williams and Johnson.[11] In this paper we reformulate for the class of semisimilar solutions so that a wide variety of special cases can be studied. At the same time, as may be anticipated, the reformulation also pinpoints the key factor that is reasonable for the singular behavior of the boundary layer. At least for the special class studied, the effect of unsteadiness is clearly displayed, and a criterion of nonsingular solution is identified. Two examples of unsteady and nonseparating boundary layer with closed recirculating bubbles are computed, without the need for special differencing schemes. The thermal boundary layers are then calculated.

II. Reformulation of the Semisimilar Boundary-Layer Equation

For an incompressible flow over a fixed impermeable solid wall, the unsteady two-dimensional boundary-layer equations are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial U_e}{\partial t}+U_e\frac{\partial U_e}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2}$$

with boundary conditions

$$u(x,0,t)=v(x,0,t)=0,\quad \lim_{y\to\infty} u(x,y,t)=U_e(x,t) \tag{3}$$

where u,v and x,y are the velocity components and coordinates parallel and perpendicular to the wall, respectively, t is time, ν is the kinematic viscosity, and $U_e(x,t)$ is a specified unsteady velocity distribution outside the boundary layer. We shall restrict ourselves in looking for the semisimilar solutions in which the number of independent variables is reduced from three to two. To achieve this, we introduce the new independent variables

$$\xi=\xi(x,t),\quad \eta=y/\sqrt{\nu}\delta(x,t) \tag{4}$$

and new dependent variables

$$\bar{u}=u/U_\infty,\quad \bar{v}=v/\sqrt{\nu}g(x,t) \tag{5}$$

where U_∞ is a constant with the dimension of velocity; the functions $\delta(x,t)$, with dimension $(\text{time})^{1/2}$ and $g(x,t)$ with dimension $(\text{time})^{-1/2}$ are to be determined. Also we assume that the flow outside the boundary layer can be expressed as

$$U_e(x,t)=U_\infty F(\xi) \tag{6}$$

Presented as Paper 77-684 at the AIAA 10th Fluid and Plasmadynamics Conference, Albuquerque, N. Mex., June 27-29, 1977; submitted Jan. 13, 1978; revision received June 19, 1978,

Index categories: Boundary Layers and Convective Heat Transfer—Laminar; Nonsteady Aerodynamics; Computational Methods.

*Postdoctoral Research Associate. Sibley School of Mechanical and Aerospace Engineering. Presently with AiResearch Mfg. Co., Phoenix, Ariz.

†Professor. Sibley School of Mechanical and Aerospace Engineering.

where $F(\xi)$ is to be determined from given $U_e(x,t)$.

With the new variables and functions defined by Eqs. (4-6), the momentum equation, Eq. (1), becomes

$$\frac{\partial^2 \bar{u}}{\partial \eta^2}+[a_1\eta+a_2\eta\bar{u}-a_3\bar{v}]\frac{\partial \bar{u}}{\partial \eta}-[a_4+a_5\bar{u}]\frac{\partial \bar{u}}{\partial \xi}$$

$$=-[a_4+a_5F(\xi)]\frac{\mathrm{d}F(\xi)}{\mathrm{d}\xi} \tag{7}$$

and the continuity equation, Eq. (2),

$$a_5\frac{\partial \bar{u}}{\partial \xi}-a_2\eta\frac{\partial \bar{u}}{\partial \eta}+a_3\frac{\partial \bar{v}}{\partial \eta}=0 \tag{8}$$

where

$$a_1=\delta\frac{\partial \delta}{\partial t},\quad a_2=U_\infty\delta\frac{\partial \delta}{\partial x},\quad a_3=\delta g,$$

$$a_4=\delta^2\frac{\partial \xi}{\partial t},\quad a_5=U_\infty\delta^2\frac{\partial \xi}{\partial x} \tag{9}$$

For the semisimilar solution to exist, a_1, a_2, a_3, a_4, and a_5 must be, at most, functions of ξ alone. Equations (7) and (8) are to be solved for $\bar{u}$ and $\bar{v}$, subject to the boundary conditions

$$\bar{u}(\xi,\eta=0)=\bar{v}(\xi,\eta=0)=0 \tag{10a}$$

$$\lim_{\eta\to\infty}\bar{u}(\xi,\eta)=F(\xi) \tag{10b}$$

We shall not discuss the solutions for a_1 through a_5 any further, but assume that they exist.

For the steady boundary-layer flow over a sharp leading edge with $U_e(x)$ given, a set of boundary-layer variables frequently used is, e.g., Howarth,[12]

$$\xi=x,\quad \eta=(U_\infty/2\nu x)^{1/2}y \tag{11}$$

Consequently, the steady momentum equation is reduced to the form

$$\frac{\partial^2 \bar{u}}{\partial \eta^2}+(\eta\bar{u}-a_3\bar{v})\frac{\partial \bar{u}}{\partial \eta}-2\xi\bar{u}\frac{\partial \bar{u}}{\partial \xi}=-2\xi F(\xi)\frac{\mathrm{d}F(\xi)}{\mathrm{d}\xi} \tag{12}$$

where $F(\xi)=U_e(\xi)/U_\infty$. Equation (12) can be found in many reports, for example, Blottner.[13]

Equations (7) and (12) are the momentum equations for unsteady and steady boundary-layer flows, repectively. Both equations are parabolic in the (ξ,η) plane. The coefficients of the streamwise derivative $\partial\bar{u}/\partial\xi$, (a_4+a_5u) and $2\xi\bar{u}$, play the same role in Eqs. (7) and (12), respectively, as that of the inverse of the thermal diffusivity in the heat-conduction equation. The importance of this coefficient in the heat equation is well known. For convenience of discussion we shall denote it by $k(\xi,\bar{u})$.

We now consider the special class of problems, as formulated in Williams and Johnson,[11] which admits the transformation

$$\xi=Ax/(1-Bt) \tag{13}$$

where A and B are positive constants. The time t will be restricted to $t<t_0=1/B$. For this class, the admissible functions may be chosen to be

$$\delta(x,t)=2(x/U_\infty)^{1/2},\quad g(x,t)=(U_\infty/x)^{1/2} \tag{14}$$

Using Eqs. (9) and (14) we obtain

$$a_1=0,\quad a_2=2,\quad a_3=2,\quad a_4=4\lambda\xi^2,\quad a_5=4\xi \tag{15}$$

where $\lambda=B/U_\infty A$.

With these results, the momentum equation, Eq. (7), becomes

$$\frac{\partial^2 \bar{u}}{\partial \eta^2}+(2\eta\bar{u}-2\bar{v})\frac{\partial \bar{u}}{\partial \eta}-4\xi(\lambda\xi+\bar{u})\frac{\partial \bar{u}}{\partial \xi}$$

$$=-4\xi[\lambda\xi+F(\xi)]\frac{\mathrm{d}F(\xi)}{\mathrm{d}\xi} \tag{16}$$

and the continuity equation, Eq. (8), becomes

$$2\xi\frac{\partial \bar{u}}{\partial \xi}-\eta\frac{\partial \bar{u}}{\partial \eta}+\frac{\partial \bar{v}}{\partial \eta}=0 \tag{17}$$

The case with

$$F(\xi)=1-\xi \tag{18}$$

had first been studied by Tani[10] and recently by Williams and Johnson.[11] Unlike the present approach, they formulated the problem using streamfunction as the sole dependent variable. Tani[10] considered this flow as related to the unsteady flows in a diffuser or on an airfoil in which the angle of divergence or angle of attack varies with time, and solved the problem by expanding the streamfunction in power series of ξ.

In trying to confirm the Moore-Rott-Sears (M-R-S) separation criterion (Sears and Telionis[14]), Williams and Johnson[11] formulated the equation for the streamfunction in a coordinate system moving with the point of separation, and integrated the equation using an implicit finite-difference technique similar to that outlined by Blottner.[13] The movement of the separation is not known at the outset and is determined by iteration. They reported that as ξ approaches its separation value ξ_0, the velocity profiles in the moving coordinate are seen to approach a separation velocity profile characterized by the simultaneous vanishing of the shear and the velocity at a point within the boundary layer. The different values of λ, ξ_0, and $\bar{u}_s$ (the value of $\bar{u}$ at the separation point) as compiled from their reports, are shown in Table 1. From Table 1 it is interesting to observe that the separation point satisfies the condition

$$k(\xi_0,\bar{u}_s)=4\xi_0(\lambda\xi_0+\bar{u}_s)=0 \tag{19}$$

The correlation equation (19) is highly suggestive. Since $k(\xi,\bar{u})$ acts as the effective inverse thermal diffusivity, it is safe to expect that the boundary-layer solution should behave normally in the domain with $k(\xi,\bar{u})>0$. If beyond a certain ξ there is $k(\xi,\bar{u})<0$ for all η, the solution must quickly break down. The correlation equation (19), however, suggests that as soon as $k=0$ at *some interior point* the instability tends to stop the numerical calculation at once. In the degenerate case of steady boundary layer over a fixed wall, where $k=4\xi\bar{u}$, we observe that $k=0$ at one point (the wall $\eta=0$) along each $\xi=$ constant is not sufficient to cause instability, but $k=0$ in a strip of infinitesimal width in the η direction ($\bar{u}=0$ and $\partial\bar{u}/\partial\eta=0$ at a point) will give rise to a Goldstein singularity which blocks the flow from continuing downstream. In the example that leads to Eq. (19), the M-R-S condition is satisfied and we have indeed also an infinitesimal strip in which $k=0$. For more general cases, however, the condition $k=0$ at some interior point is not obviously identical to the M-R-S condition. Further analysis confirming the M-R-S condition, including the occurrence of a Goldstein singularity for the semisimilar boundary layer, can be found in Shen.[16]

Table 1 Values of λ, ξ_0 and $\bar{u}_s$ at the separation point

λ	ξ_0	$\bar{u}_s$
0	0.117	0
0.5	0.161	−0.0805
1.0	0.221	−0.021

III. Unsteady Boundary Layer with Closed Recirculating Bubble

Let us consider the following two cases of external flow

Case I

$$F(\xi)=1-A_1\xi\cos T_1\xi \tag{20}$$

Case II

$$F(\xi)=1-[A_2\xi/(1+T_2\xi^2)] \tag{21}$$

where ξ is defined by Eq. (13) and A_1, T_1, A_2, and T_2 are arbitrary positive constants. For small ξ (corresponding to small x), both cases reduce to the unsteady linearly retarded flow studied by Tani[10] and Williams and Johnson.[11] For large ξ, case I corresponds to an amplified oscillatory flow and catastrophic separation is expected to occur somewhere downstream; case II asymptotically becomes a uniform flow.

Equations (16) and (17), subject to the boundary condition given by Eq. (10) and $F(\xi)$ given by either Eq. (20) or Eq. (21), are solved by using the Crank-Nicolson scheme as outlined by Blottner.[13] Throughout the entire computation, η intervals are kept constant. Shortly before the detached point and throughout the entire reversed flow region the ξ intervals are many times reduced to insure that the instability is not incidentally suppressed by taking too large a step size.

The results presented here are computed by using $\lambda=1$ and $\lambda=2$ for cases I and II, respectively, and $A_1=3.10$, $T_1=8.64$, $A_2=5.650$, and $T_2=160$. For larger values of A_1 and A_2, e.g., 3.5 and 9.5, respectively, the computations pass smoothly through the point of zero wall shear and in fact break down at the point where $k=0$ at some interior point.

Figures 1a and b show the velocity profiles for case I using ξ as parameter. For any instant of time, say t_1, $t_1<1/B$ the curves on these two figures show the instantaneous velocity profiles at the locations

$$x=[(1-Bt_1)/A]\xi \tag{22}$$

The instantaneous velocity distribution of the flowfield changes smoothly from a Blasius profile at the leading edge ($\xi=0$), through the point of zero wall shear, into a region of partially reversed flow near the wall with negative wall shear, then back to a normal boundary-layer-velocity profile. For a fixed station, say x_1, the curves show the variation of the velocity profile as a function of time. Figures 2a and b show the velocity profiles for case II. The streamfunction, defined by

$$u=\frac{\partial\psi}{\partial y},\quad v=-\frac{\partial\psi}{\partial x} \tag{23}$$

is found by

$$\psi=\int_0^y u(x,s,t)\,\mathrm{d}s \tag{24}$$

where $\psi(x,0,t)$ is set to zero. The following relation can be easily established

$$\bar{\psi}=\frac{\psi}{2\sqrt{\nu x U_\infty}}=\int_0^\eta \bar{u}(\xi,s)\,\mathrm{d}s \tag{25}$$

The instantaneous streamlines, $\bar{\psi}=$constant, for cases I and II are shown by solid line in Figs. 3 and 4, respectively. In Figs. 3a and 4a, curve 1 is the distribution of Eq. (20) and Eq. (21), respectively, while curve 2 is that of Eq. (18) included for comparison. In Fig. 3b, the values of $\bar{\psi}$ for the three closed streamlines are, from outside inward, −.005, −.008, and −.0095; in Fig. 4b, the corresponding values of $\bar{\psi}$ are −.002, −.004, and −.005.

Using Eqs. (4) and (14), we have for fixed t

$$y/x=2\eta/\sqrt{R_x} \tag{26}$$

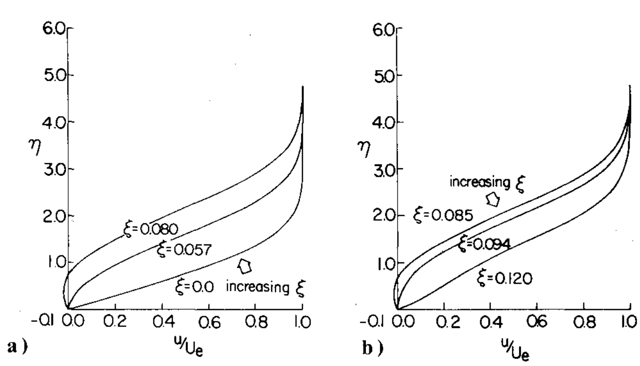

Fig. 1 Velocity profiles of the unsteady boundary layer with $U_e(x,t)/U_\infty$ given by Eq. (20).

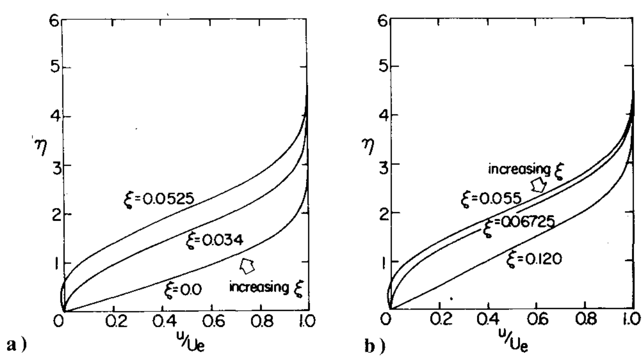

Fig. 2 Velocity profiles of the unsteady boundary layer with $U_e(x,t)/U_\infty$ given by Eq. (21).

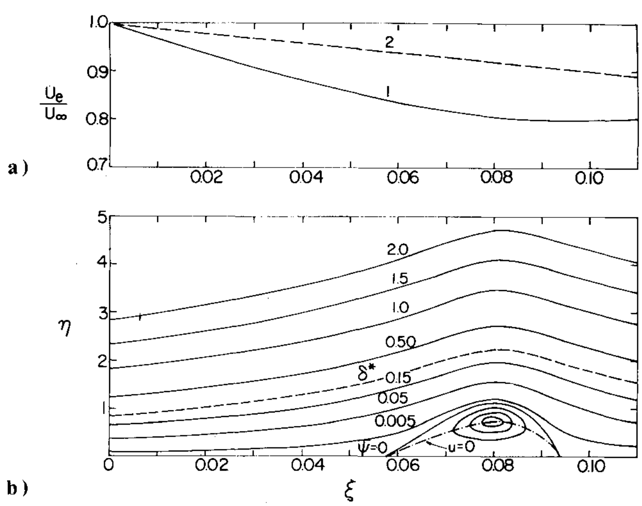

Fig. 3 Streamlines $\bar{\psi}$ and nondimensional displacement thickness δ^* for case I.

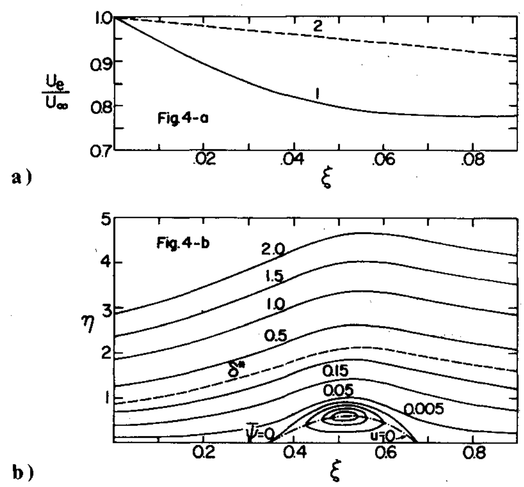

Fig. 4 Streamlines $\bar{\psi}$ and nondimensionalized displacement thickness δ^* for case II.

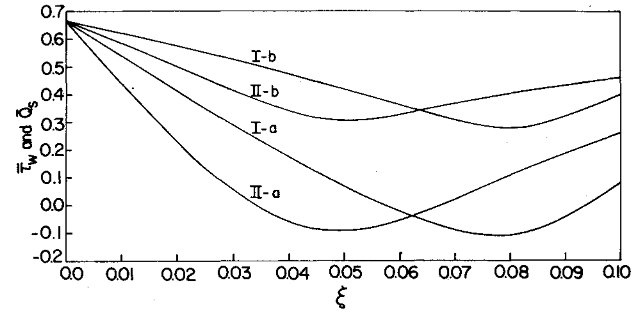

Fig. 5 Nondimensionalized skin frictions $\bar{\tau}_w$ and heat-transfer coefficients $\bar{Q}_s$.

where R_x is the Reynolds number based on the length from the leading edge. From the results we obtained, the height of these bubbles was inversely proportional to $\sqrt{R_x}$. Therefore, as $\nu \to 0$, i.e., $R_x \to \infty$, the height of these bubbles tends to zero. There is no separation and the boundary-layer equation should still be valid both inside and outside these bubbles. Note that the word "bubbles" here is loosely used to denote the presence of reverse flow. As plotted in the (ξ,η) plane, the streamline $\bar{\psi}=$constant does not exactly indicate the instataneous directions of the particle movement.

The displacement thickness is defined by

$$\delta_I(x,t) = \int_0^\infty (1-u/U_e)\,\mathrm{d}y \tag{27}$$

A nondimensionalized displacement thickness can be expressed as

$$\delta^*(\xi) = \delta_I(x,t)\frac{2\sqrt{U_\infty}}{\sqrt{\nu x}} = \lim_{h\to\infty}\int_0^h \left[1-\frac{\bar{u}(\xi,s)}{F(\xi)}\right]\mathrm{d}s \tag{28}$$

$\delta^*(\xi)$ for cases I and II are shown by broken lines in Figs. 3 and 4, respectively. Lines of $\delta^*(\xi)$ coincide approximately with the lines of $\bar{\psi}=0.25$ for both cases. In Figs. 3 and 4 we also show by dashed broken lines, the lines of $\bar{u}(\xi,\eta\neq 0)=0$. These lines coincide with the locus of the points of the maximum maximorum of the circulating streamlines. Figure 5 shows the well-behaved wall-shear stress, where $\tau_w \equiv (\partial u/\partial y)_{y=0}/(U_\infty^{3/2}/\sqrt{\nu x})$.

IV. Associated Thermal Boundary Layer

Let us consider the thermal boundary layer governed by the equation[15]

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \beta\frac{\partial^2 T}{\partial y^2} \tag{29}$$

where β is the thermal diffusivity and T the temperature. The heat generated by viscous dissipation is neglected, and the temperature of the free stream and the wall are assumed to be constant,

$$T(x,\ y=0,\ t) = T_w \tag{30a}$$

$$\lim_{y\to\infty} T(x,y,t) = T_\infty \tag{30b}$$

where T_w and T_∞ are constants.

Defining dimensionless temperature

$$\bar{T} = (T-T_w)/(T_\infty - T_w) \tag{31}$$

and using the semisimilar variables defined in Section II, we obtain from Eq. (31)

$$\frac{1}{Pr}\frac{\partial^2\bar{T}}{\partial\eta^2} + (2\eta\bar{u}-2\bar{v})\frac{\partial\bar{T}}{\partial\eta} - 4\xi(\lambda\xi+\bar{u})\frac{\partial\bar{T}}{\partial\xi} = 0 \tag{32}$$

where $Pr=\nu/\beta$, the Prandtl number.

The boundary conditions given by Eq. (30) become

$$\bar{T}(\xi,0)=0 \quad \lim_{\eta\to\infty}\bar{T}(\xi,\eta)=1 \tag{33}$$

Corresponding to the flows specified by Eqs. (20) and (21), Eq. (32), with $Pr=1$, is solved subject to boundary conditions, Eq. (33). The nondimensionalized heat flux to the wall

$$\bar{Q}_s = k\frac{\partial T}{\partial y}\bigg|_{y=0} \bigg/ \left[\frac{1}{2}k\sqrt{U_\infty/\nu x}(T_\infty - T_w)\right] = \frac{\partial\bar{T}}{\partial\eta}\bigg|_{\eta=0} \tag{34}$$

is shown in Fig. 5 together with the skin-friction coefficient. Here I-a and II-a are the curves for the nondimensionalized skin friction for the boundary layers of cases I and II, respectively; I-b and II-b are the nondimensionalized heat flux to the wall associated with the boundary layers of cases I and II, respectively. It is worth noting that the existence of the flat recirculating bubble does not affect the heat-transfer coefficient drastically. Consequently, the usual concept of a "Reynolds analogy" implying a proportionality between heat transfer and skin friction must be abandoned in such cases.

V. Conclusions

The semisimilar unsteady boundary-layer equation has been integrated before in terms of the streamfunction. We find it preferable to retain the velocity components as the dependent variables so that the numerical solution can proceed along established paths, e.g., Blottner.[13] This formulation also reveals clearly that the essential change due to unsteadiness is to introduce an effective convective speed $k(\xi,\bar{u})$, instead of the streamwise velocity component $\bar{u}$, as the coefficient of $\partial\bar{u}/\partial\xi$ in the momentum equation, Eq. (7).

It follows naturally that a unified separation criterion for both the steady and unsteady cases might be possible if formulated in terms of k. From available evidences, we suggest the criterion as $k=0$ at some interior point of the fluid. It includes Prandtl's criterion for the steady separation as a special case and, in the special cases treated by Tani[10] and Williams and Johnson,[11] agrees with the much discussed M-R-S condition. For further assessment we refer to Shen.[16]

The specific examples of unsteady boundary layer with a bubble of recirculating flow are constructed to verify that no special differencing scheme is needed when $k>0$. The vanishing wall shear and local flow reversal are seen to play no part in unsteady separation. The associated thermal boundary layer shows no unusual feature, and the local heat-transfer coefficient does not reflect the presence of the reverse flow. Thus it would not be possible to detect local flow reversal via "Reynolds' analogy."

Acknowledgment

This work was partially supported by the U.S. Air Force Office of Scientific Research under Grant AFOSR-74-2659. The authors wish to thank C. A. Hieber and S. K. Kim for their discussions on the numerical integrations of boundary-layer equations.

References

[1]Goldstein, S., "On Laminar Boundary-Layer Flow Near a Position of Separation," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 1, 1948, pp. 43-69.

[2]Catherall, D. and Mangler, K. W., "The Integration of the Two-Dimensional Laminar Boundary-Layer Equations Past the Point of Vanishing Skin-Friction," *Journal of Fluid Mechanics*, Vol. 26, Sept. 1966, pp. 163-182.

[3]Klineberg, J. M. and Steger, J. L., "On Laminar Boundary-Layer Separation," AIAA Paper 74-94, 1974.

[4]Rott, N., "Unsteady Flow in the Vicinity of a Separation Point," *Quarterly of Applied Mathematics*, Vol. 13, Jan. 1956, pp. 444-451.

[5]Sears, W. R., "Some Recent Developments in Airfoil Theory," *Journal of the Aerospace Sciences*, Vol. 23, 1956, pp. 490-499.

[6]Moore, F. K., "On the Separation of the Unsteady Laminar Boundary Layer," *Boundary Layer Research*, edited by H. Görtler, Springer, Berlin, 1958, pp. 296-310.

[7]Shen, S. F. and Nenni, J. P., "Asymptotic Solution of the Unsteady Two-Dimensional Incompressible Boundary Layer and Its Implication on Separation," *Unsteady Aerodynamics*, Vol. 1, edited by B. B. Kinney, July 1975, pp. 245-259.

[8]Telionis, D. P., Tsahalis, D. Th., and Werle, M. J., "Numerical Investigation of Unsteady Boundary Layer Separation," *Physics of Fluids*, Vol. 16, July 1973, pp. 968-973.

[9]Phillips, J. H. and Ackerberg, R. C., "A Numerical Method for Integrating the Unsteady Boundary-Layer Equations When There are Regions of Back-Flow," *Journal of Fluid Mechanics*, Vol. 58, May 1973, pp. 561-579.

[10]Tani, I., "An Example of Unsteady Laminar Boundary Layer Flow," Inst. Univ. of Tokyo, Rept. 331, 1958 and *Boundary Layers*, IUTAM-Symposium, Freiburg 1957, Berlin, 1958, p. 347.

[11]Williams, J. C., IV and Johnson, W. D., "Semisimilar Solutions to Unsteady Boundary-Layer Flows Including Separation," *AIAA Journal*, Vol. 12, Oct. 1974, pp. 1388-1393.

[12]Howarth, L., "On the Solution of the Laminar Boundary-Layer Equations," *Proceedings of the Royal Society of London*, Vol. A164, Feb. 1938, pp. 547-579.

[13]Blottner, F. G., "Finite-Difference Methods of Solution of the Boundary-Layer Equations," *AIAA Journal*, Vol. 8, Feb. 1970, pp. 193-205.

[14]Sears, W. R. and Telionis, D. P., "Unsteady Boundary Layer Separation," in *Fluid Dynamics of Unsteady Three-Dimensional and Separated Flows*, edited by F. J. Marshall, Purdue University, Lafayette, Ind., 1972, pp. 245-305.

[15]Rosenhead, L., ed., *Laminar Boundary Layers*, Oxford University Press, New York, 1963, p. 406.

[16]Shen, S. F., "Unsteady Separation According to the Boundary Layer Equation," *Advances in Applied Mechanics*, Vol. 18, 1978, in press.